

# GREEN'S $\mathcal{J}$ -CLASSES AND SUBDUCTION CLASSES IN FINITE TRANSFORMATION SEMIGROUPS

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**ABSTRACT.** We establish the connection between Green's  $\mathcal{J}$ -classes and the subduction equivalence classes defined on the image sets of an action of a semigroup. The construction of the skeleton order (on subduction equivalence classes) is shown to depend in a functorial way on transformation semigroups and surjective morphisms, and to factor through the  $\leq_{\mathcal{L}}$ -order and  $\leq_{\mathcal{J}}$ -order on the semigroup and through the inclusion order on image sets. For right regular representations, the correspondence between the  $\mathcal{J}$ -class order and the skeleton is one of isomorphism.

## 1. INTRODUCTION

The holonomy decomposition algorithm [13, 14, 7, 5, 8, 12, 3] is a wreath product decomposition theorem for finite transformation semigroups. It yields a Krohn-Rhodes decomposition by the detailed analysis of the semigroup action on all subsets of the state set which occur as images under the semigroup action. One of the main tools of this analysis is the *subduction* preorder relation defined on the set of images of the members of the semigroup considered as mappings. (See below for precise definitions.) Green's preorders give ample information about the semigroup's internal structure, while subduction captures details of the semigroup action. Therefore, the natural question arises: *What is the connection between the Green's relations and the subduction relation?* More generally, by aiming to describe the connection between a semigroup action and the internal structure of semigroup itself we may get information on what transformation representations are possible for an abstract semigroup.

**1.1. Notation.**  $(X, S)$  is a *transformation semigroup* with  $S$  acting faithfully on the *state set*  $X$  if  $S$  is a subsemigroup of the (right) full transformation semigroup  $\mathcal{T}(X)$  of all mappings on a set  $X$ . For  $x \in X, s \in S$ , the result of the action is written  $x^s$ . The action can be extended to subsets of  $X$ , if  $P \subseteq X$  and  $s \in S$  then  $P^s = \{x^s \mid x \in P\}$ . The *image* of a transformation  $s$  is defined by  $\lambda(s) = X^s$ , and we can also say that  $\lambda(s)$  is the image of  $X$  under  $s$ .  $S^1$  is the monoid obtained by adjoining the identity on  $X$  to the semigroup  $S$ , if it is not a member of  $S$ , otherwise  $S^1 = S$ .

$(A, \leq)$  is a *preorder* (sometimes called a 'quasi-order') if  $\leq$  is a reflexive and transitive relation on the set  $A$ . For a preorder, there exists an equivalence relation  $(A, \equiv)$  defined by  $a \equiv b \iff a \leq b$  and  $b \leq a$ , and an induced partial order on the equivalence classes  $(A/\equiv, \leq)$ . The surjective map  $A \twoheadrightarrow A/\equiv$  is denoted by  $\eta^1$ .

The classical Green's relations  $\leq_{\mathcal{R}}, \leq_{\mathcal{L}}, \leq_{\mathcal{J}}$  and  $\leq_{\mathcal{H}}$ , on any semigroup  $S$  are the preorders:

$$t \leq_{\mathcal{R}} s \iff tS^1 \subseteq sS^1,$$

$$\begin{aligned} t \leq_{\mathcal{L}} s &\iff S^1 t \subseteq S^1 s, \\ t \leq_{\mathcal{J}} s &\iff S^1 t S^1 \subseteq S^1 s S^1 \end{aligned}$$

and  $\leq_{\mathcal{H}}$  is the intersection of the  $\leq_{\mathcal{L}}$  and  $\leq_{\mathcal{R}}$  relations. Then  $\mathcal{R}, \mathcal{L}, \mathcal{J}$  and  $\mathcal{H}$  denote the equivalence relations arising from the preorders  $\leq_{\mathcal{R}}, \leq_{\mathcal{L}}, \leq_{\mathcal{J}}$  and  $\leq_{\mathcal{H}}$ , respectively, and in each case the equivalence classes carry the induced partial order. The equivalence relation  $\mathcal{D}$  on  $S$  is the composite of  $\mathcal{L}$  and  $\mathcal{R}$ , which commute. (For standard definitions and elementary properties see for instance [1, 10, 9]). In the finite case, the  $\leq_{\mathcal{J}}$  and  $\leq_{\mathcal{D}}$ , and thus the  $\mathcal{J}$  and  $\mathcal{D}$  relations coincide. Here we only consider finite transformation semigroups. However, we still use the  $\mathcal{J}$ -ordering of  $\mathcal{J}$ -classes, as by definition  $\mathcal{D}$  is an equivalence relation that does not necessarily come from a preorder in the general case (see e.g. [10]). Though currently there is no infinite version of the holonomy theorem using subduction, we would like to keep the proofs of results given here compatible with possible future developments.

## 2. SUBDUCTION RELATION

For a transformation semigroup  $(X, S)$  the set  $\mathcal{I}(X) = \{\lambda(s) \mid s \in S^1\}$  is the *image set* of the semigroup action. Note  $X = \lambda(1)$  is always in  $\mathcal{I}(X)$ .

**Definition 2.1** (Subduction). For  $P, Q \in \mathcal{I}(X)$

$$P \subseteq_S Q \iff \exists s \in S^1 \text{ such that } P \subseteq Q^s.$$

So  $P$  is a subset of  $Q$  or it is a subset of some image of  $Q$  under the semigroup action.

**Lemma 2.2.** 1.  $(\mathcal{I}(X), \subseteq_S)$  is a preorder.

2. If  $P \subseteq_S Q$  and  $Q \subseteq_S P$  then  $|P| = |Q|$ .

*Proof.* Obviously  $\subseteq_S$  is reflexive, since  $P \subseteq P^1$ . It is transitive, since if  $P \subseteq Q^{s_1}$  and  $Q \subseteq R^{s_2}$  then  $P \subseteq R^{s_2 s_1}$ . For (2), there exists  $s \in S^1$  with  $P \subseteq Q^s$ , so  $Q$  has cardinality at least that of  $P$ . By symmetry, it follows that  $P$  and  $Q$  have the same cardinality.  $\square$

Therefore, we can naturally define by mutually subduced subsets an equivalence relation on  $\mathcal{I}(X)$ , denoted by  $\equiv_S$ .

**Corollary 2.3.**  $(\mathcal{I}(X)/\equiv_S, \subseteq_S)$  is a partial order.

One calls the subduction class ordering of  $(X, S)$  its *skeleton ordering* for  $(X, S)$ . This structure provides the scaffolding for a holonomy decomposition since subduction equivalent subsets have isomorphic holonomy permutation groups, so only one copy of these groups is needed per class in the decomposition [13, 14, 7, 5, 8, 3]. For the holonomy decomposition the skeleton order is extended by using the *extended image set*  $\mathcal{I}^+(X) = \mathcal{I}(X) \cup \{\{x\} : x \in X\}$  which includes any singletons that do not occur as images. This could potentially result in additional minimal equivalence classes for these singletons being adjoined to the skeleton.

## 3. $\mathcal{J}$ -CLASSES AND SKELETON CLASSES

We can establish connection between the induced classes of two preorders through a given preorder preserving map. First, we describe the situation for preorders in general.

**Lemma 3.1.** *Let  $(A, \leq_1)$  and  $(X, \leq_2)$  be preorders and  $f : A \rightarrow X$  a function respecting preordering. Then,*

- (1)  *$f$  induces an order-preserving map  $\bar{f} : (A/\equiv_1, \leq_1) \rightarrow (X/\equiv_2, \leq_2)$ , and the following diagram commutes.*

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ \downarrow \eta_1^\natural & & \downarrow \eta_2^\natural \\ A/\equiv_1 & \xrightarrow{\bar{f}} & X/\equiv_2 \end{array}$$

- (2) *The kernel, the equivalence relation induced on  $A$  by  $\eta_1^\natural \circ \bar{f} = f \circ \eta_2$  is not finer than  $\equiv_1$ .*

*Proof.* (1) Let  $\bar{f}$  denote the function taking the  $\equiv_1$ -class of any  $a \in A$  to the  $\equiv_2$ -class of  $f(a)$ . If  $a \equiv_1 b$  then by definition  $a \leq_1 b$  and  $b \leq_1 a$ . Since  $f$  respects preordering,  $f(a) \leq_2 f(b)$  and  $f(b) \leq_2 f(a)$ , therefore  $f(a) \equiv_2 f(b)$ . It follows that  $\bar{f}$ , given by  $\bar{f}([a]_1) = [f(a)]_2$ , is well-defined and order preserving.

(2) Since  $\bar{f}$  is a function, the inverse image of an  $\equiv_2$ -class consists of  $\equiv_1$ -classes. Hence the inverse image of this in  $A$  is the union of  $\equiv_1$ -equivalence classes.  $\square$

**Remark:** It is important to notice that  $a <_1 b$  does not exclude the possibility of  $f(a) \equiv_2 f(b)$ . Moreover, even if neither  $a \leq_1 b$  nor  $b \leq_1 a$  holds one might still have  $f(a) <_2 f(b)$  or  $f(a) \equiv_2 f(b)$ .

Now we have two preorders:  $\leq_{\mathcal{J}}$  on  $S$  and subduction  $\subseteq_S$  on  $\mathcal{I}(X)$ . Next we show that the surjective function  $\lambda$  respects preordering. For the weaker case, it is a basic fact that  $a \mathcal{L} b \implies \lambda(a) = \lambda(b)$ . However,  $\mathcal{J}$ -related elements can have different images. For instance, in the full transformation semigroups on  $n$  points, all constant maps are  $\mathcal{J}$ -equivalent to each other.

**Lemma 3.2.** *For any transformation semigroup  $(X, S)$  and any  $a, b \in S$ , we have*

$$a \leq_{\mathcal{L}} b \implies \lambda(a) \subseteq \lambda(b).$$

$$a \leq_{\mathcal{J}} b \implies \lambda(a) \subseteq_S \lambda(b).$$

*That is,  $\lambda$  maps the  $\mathcal{L}$ -preorder to the inclusion partial order and maps the  $\mathcal{J}$ -preorder to the subduction preorder. Moreover,  $\lambda$  induces a surjective map from  $S^1$  in each case.*

*Proof.* The first assertion is well-known: If  $a \leq_{\mathcal{L}} b$  then  $a = sb$  for some  $s \in S^1$ . Thus  $\lambda(a) = X^a = X^{sb} = (X^s)^b \subseteq X^b = \lambda(b)$ .

For the second, if  $a \leq_{\mathcal{J}} b$  then there exist  $s, t \in S^1$  such that  $a = sbt$ ,

$$\lambda(a) = \lambda(sbt) = \lambda(sb)^t \subseteq \lambda(1b)^t = \lambda(b)^t,$$

therefore  $\lambda(a) \subseteq_S \lambda(b)$ . Obviously  $\lambda$  maps  $S^1$  surjectively onto  $\mathcal{I}(S) = \{\lambda(s) : s \in S\}$ , hence onto the preorder  $(\mathcal{I}(X), \subseteq_S)$  which has the same underlying set.  $\square$

**Theorem 3.3.** *For a transformation semigroup  $(X, S)$ , there is a surjective order-preserving map  $\bar{\lambda}_S$  from the partial order of  $\mathcal{J}$ -classes  $(S^1/\mathcal{J}, \leq_{\mathcal{J}})$ , onto the partial order of subduction classes  $(\mathcal{I}(X)/\equiv_S, \subseteq_S)$ . The inverse image of a subduction equivalence class is a union of  $\mathcal{J}$ -classes.*

*Proof.*  $\lambda$  is a surjective, and is a preorder morphism from the Green's  $\mathcal{J}$  preorder to the subduction preorder by Lemma 3.2, therefore by using Lemma 3.1(1), the induced map  $\bar{\lambda}$ , which we shall denote  $\bar{\lambda}_S$  to distinguish it from the map in the next result, is a surjective order-preserving map. By Lemma 3.1(2), the inverse image of a subduction class corresponds to a union of  $\mathcal{J}$ -classes.  $\square$

Similarly, generalizing the basic fact mentioned above, we have

**Theorem 3.4.** *For a transformation semigroup  $(X, S)$ , there is a surjective order-preserving map  $\bar{\lambda}$  from the  $\mathcal{L}$ -class order for  $S^1$  onto the inclusion partial order on  $\mathcal{I}(X)$ . The inverse image of an image set is a union of  $\mathcal{L}$ -classes.*

Putting these facts together, it follows that

**Theorem 3.5.** *For any transformation semigroup  $(X, S)$ , there is a commutative diagram of surjective order-preserving morphisms:*

$$\begin{array}{ccc}
 (S^1, \leq_{\mathcal{L}}) & & \\
 \downarrow /_{\mathcal{L}} & \searrow \lambda & \\
 (S^1/\mathcal{L}, \leq_{\mathcal{L}}) & \xrightarrow{\bar{\lambda}} & (\mathcal{I}(X), \subseteq) \\
 \downarrow /_{\mathcal{J}} & & \downarrow /_{\equiv_S} \\
 (S^1/\mathcal{J}, \leq_{\mathcal{J}}) & \xrightarrow{\bar{\lambda}_S} & (\mathcal{I}(X)/\equiv_S, \subseteq_S)
 \end{array}$$

**Corollary 3.6.** *For the right regular representation  $(S^1, S)$ :*

- (1) *The  $\mathcal{J}$ -class order and the subduction order are isomorphic.*
- (2) *The  $\mathcal{L}$ -class order and the inclusion order on image sets  $\mathcal{I}(X)$  are isomorphic.*

*Proof.* (1) By Lemma 3.2, it suffices to show that  $\lambda(a) \subseteq_S \lambda(b) \implies a \leq_{\mathcal{J}} b$ . By definition of subduction  $\lambda(a) \subseteq \lambda(b)^t$  for some  $t \in S^1$ . Since  $X = S^1$  we can write  $\lambda(a)$  as  $(S^1)^a$ , or by shifting notation from semigroup action to semigroup multiplication, simply as  $S^1 a$ . Therefore,

$$S^1 a \subseteq S^1 b t \implies S^1 a S^1 \subseteq S^1 b t S^1 \subseteq S^1 b S^1 \implies a \leq_{\mathcal{J}} b.$$

It follows that, if  $\lambda(a) \not\subseteq_S \lambda(b)$  then  $a \mathcal{J} b$  does not hold. Thus  $\bar{\lambda}_S$  is injective, hence bijective.

(2) More simply for the  $\mathcal{L}$ -order,  $\lambda(a) \subseteq \lambda(b)$  in the case of the right regular representation is just  $(S^1)^a \subseteq (S^1)^b$ , i.e.,  $S^1 a \subseteq S^1 b$ , which is the definition of  $a \leq_{\mathcal{L}} b$ . Hence,  $\lambda(a) \subseteq \lambda(b)$  implies  $a \leq_{\mathcal{L}} b$ . By Lemma 3.2 for the  $\leq_{\mathcal{L}}$ -preorder, the converse holds. It follows that if  $\lambda(a) \neq \lambda(b)$  then it cannot be that  $a \mathcal{L} b$ , hence  $\bar{\lambda}$  is injective, and hence bijective as well.  $\square$

In the case of the right regular representation this says that the horizontal mapping in Theorem 3.5 are order isomorphisms.

Both the  $\mathcal{J}$ -class order and the skeleton capture information about the structure of the semigroup, therefore surjective homomorphisms should respect them.

**Theorem 3.7** (Functoriality). *Suppose  $\varphi : (X, S) \twoheadrightarrow (Y, T)$  is a surjective morphism of transformation semigroups such that if  $1 \in S$  then  $\varphi(1)$  is the identity on*

$Y$ . Then  $\varphi$  induces a natural mapping of the commutative diagram for  $(X, S)$  as in Theorem 3.5, to the commutative diagram for  $(Y, T)$ .

*Proof.* A surjective map of semigroups induces a surjective map of the  $\leq_{\mathcal{L}}$  and  $\leq_{\mathcal{J}}$  pre-orders and orderings (as well as for  $\leq_{\mathcal{R}}$  and  $\leq_{\mathcal{H}}$ ).  $\varphi$  also induces a surjective map from  $\mathcal{I}(X)$  onto  $\mathcal{I}(Y)$ , and subduction in the source implies subduction in the target since  $P \subseteq Q^s$  implies  $\varphi(P) \subseteq \varphi(Q^s) = \varphi(Q)^{\varphi(s)}$ , hence the subduction relation is respected, and the result follows.  $\square$

#### 4. EXAMPLES

We present a few examples to illustrate the connection between the  $\mathcal{J}$ -class order and the skeleton. The partial orders are displayed as Hasse diagrams. Shaded clusters of  $\mathcal{J}$ -classes are mapped to a single subduction class.

**Example 1** (Simple collapsing of a chain). Let  $X = \{1, 2, 3\}$ ,  $t_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}$ ,  $t_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix}$ ,  $t_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$  and  $M$  the monoid  $\{1, t_1, t_2, t_3\}$ , so  $(X, M)$  is a transformation monoid on 3 points. The principal two-sided ideals are:

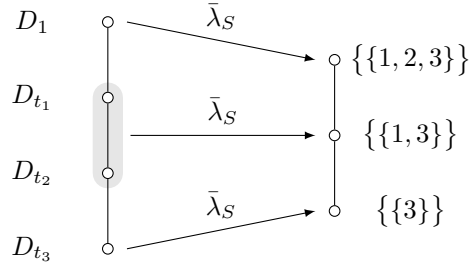
$$M1M = M$$

$$Mt_1M = \{t_1, t_2, t_3\}$$

$$Mt_2M = \{t_2, t_3\}$$

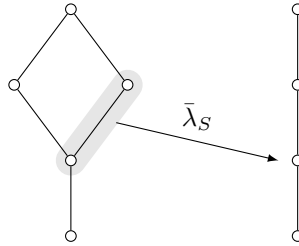
$$Mt_3M = \{t_3\},$$

therefore  $t_3 <_{\mathcal{J}} t_2 <_{\mathcal{J}} t_1 <_{\mathcal{J}} 1$  and all elements form a singleton  $\mathcal{J}$ -class on their own.  $\mathcal{I}(X) = \{\{1, 2, 3\}, \{1, 3\}, \{3\}\}$  defines the subduction classes.

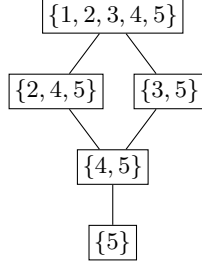


A simple linear order is mapped to a shorter linear order, since  $\lambda(t_1) = \lambda(t_2) = \{1, 3\}$ .

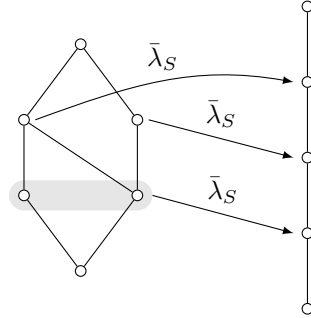
**Example 2.** More general collapsing (a usual motif) for a transformation monoid on 3 points,  $M = \{1, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}\}$ .



The right regular transformation representation of  $M$  can be encoded as  $M' = \{1, (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 4 & 4 & 5 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 3 & 5 & 5 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & 5 & 5 \end{smallmatrix}), (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 5 & 5 & 5 & 5 \end{smallmatrix})\}$ . Its skeleton is isomorphic to the  $\mathcal{J}$ -class order of  $M$ .



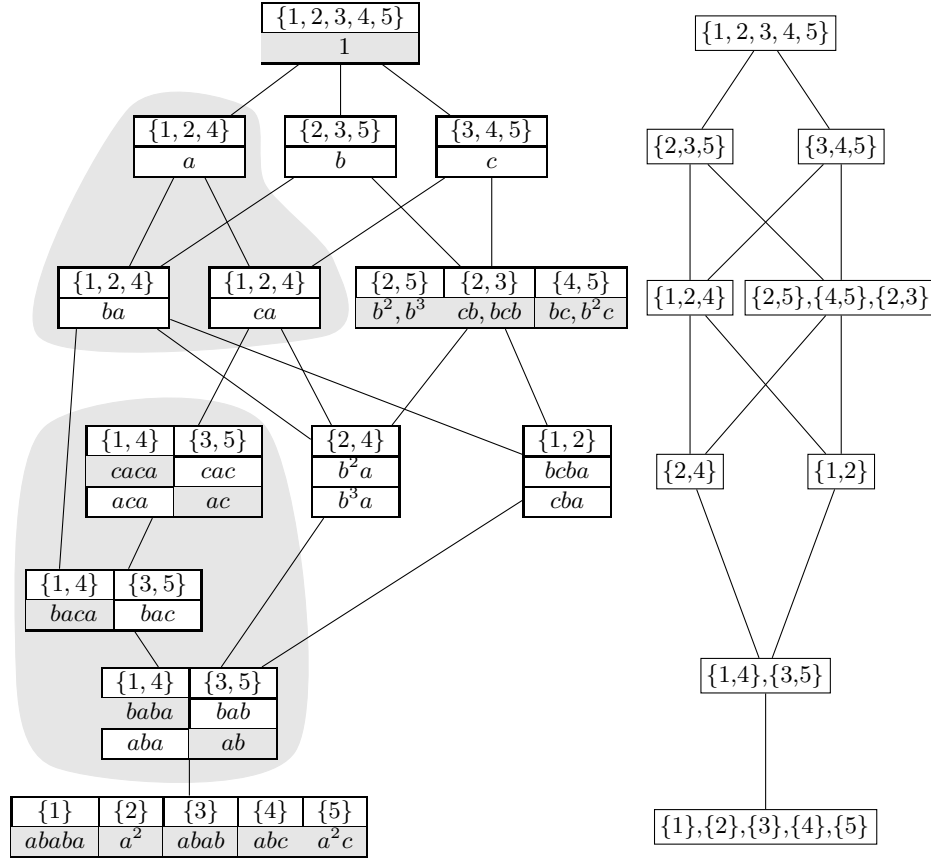
**Example 3.**  $M$  monoid generated by  $a = (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 1 & 1 & 4 \end{smallmatrix})$  and  $b = (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 4 \end{smallmatrix})$ . In  $M$ ,  $a$  and  $b$  are  $\leq_{\mathcal{J}}$ -incomparable, but  $\lambda(a) \subset_S \lambda(b)$ . This is so as there is no solution for the equation  $b = sat$  or  $a = sbt$  for  $s, t \in M$ , although  $\lambda(a) \subset \lambda(b)$ .



This shows that the subduction order may contain new relations beyond those induced by collapsing nodes of the  $\mathcal{J}$ -order diagram. Consequently, the length of a longest  $\mathcal{J}$ -chain is not an upper bound for the height of the skeleton.

So far the  $\mathcal{J}$ -class orders were all lattices, but this is not true in general, therefore we have to look at a monoid with more inner structure.

**Example 4** (Nonlinear, non-lattice skeleton). Let  $a = (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 1 & 2 & 4 \end{smallmatrix})$ ,  $b = (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 3 & 2 \end{smallmatrix})$ ,  $c = (\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 5 & 4 \end{smallmatrix})$  and  $M = \langle a, b, c \rangle$ .  $|M| = 31$ ,  $|\mathcal{I}(X)| = 16$ , number of  $\mathcal{D}$ -classes is 13, and the number of skeleton classes is 9. On the left the  $\mathcal{D}$ -class picture is drawn. On top of each  $\mathcal{L}$ -class (drawn vertically) the corresponding image is displayed.  $\mathcal{H}$ -classes with an idempotent are shaded. The grey background blobs indicate  $\mathcal{D}$ -classes that are collapsed into one subduction class. On the right the skeleton order is drawn. It is nonlinear and it is not a lattice. The boxes indicate subduction equivalence classes.



The skeleton also contains nonsingleton subduction equivalence classes.

## 5. CONCLUSION

Working towards a simplified and elementary description of the holonomy decomposition, we clarified the connection between the  $\mathcal{J}$ -classes of a semigroup and the subduction classes of a transformation representation of the same semigroup. We showed how the partial order of  $\mathcal{J}$ -classes constrains the image relations in the possible (faithful) actions of the semigroup. Therefore, these results may also be useful for investigating or enumerating the possible action representations of a semigroup. Theorem 3.3 suggests that the holonomy decomposition might be made functorial, or nearly functorial, for a suitable category of transformation semigroups and surjective morphisms since the skeleton order is.

For calculating and checking the examples we used the GAP [6] packages SEMIGROUPS [11], SGPDEC [4] and SGPVIZ[2].

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